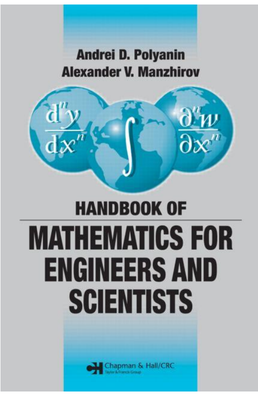

Handbook of Mathematics for Engineers and Scientists

Andrei D. Polyanin, Alexander V. Manzhirov

Elementary Geometry

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Andrei D. Polyanin, Alexander V. Manzhirov
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Chapter 3
Elementary Geometry

3.1. Plane Geometry

3.1.1. Triangles

3.1.1-1. Plane triangle and its properties.

1°. A *plane triangle*, or simply a *triangle*, is a plane figure bounded by three straight line segments (*sides*) connecting three noncollinear points (*vertices*) (Fig. 3.1*a*). The smaller angle between the two rays issuing from a vertex and passing through the other two vertices is called an (*interior*) *angle* of the triangle. The angle adjacent to an interior angle is called an *external angle* of the triangle. An external angle is equal to the sum of the two interior angles to which it is not adjacent.

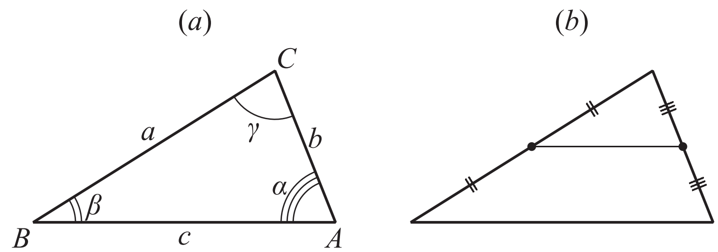

Figure 3.1. Plane triangle (*a*). Midline of a triangle (*b*).

A triangle is uniquely determined by any of the following sets of its parts:

1. Two angles and their included side.
2. Two sides and their included angle.
3. Three sides.

Depending on the angles, a triangle is said to be:

1. *Acute* if all three angles are acute.
2. *Right* (or *right-angled*) if one of the angles is right.
3. *Obtuse* if one of the angles is obtuse.

Depending on the relation between the side lengths, a triangle is said to be:

1. *Regular* (or *equilateral*) if all sides have the same length.
2. *Isosceles* if two of the sides are of equal length.
3. *Scalene* if all sides have different lengths.

2°. *Congruence tests for triangles:*

1. If two sides of a triangle and their included angle are congruent to the corresponding parts of another triangle, then the triangles are congruent.
2. If two angles of a triangle and their included side are congruent to the corresponding parts of another triangle, then the triangles are congruent.

3. If three sides of a triangle are congruent to the corresponding sides of another triangle, then the triangles are congruent.

3°. Triangles are said to be *similar* if their corresponding angles are equal and their corresponding sides are proportional.

Similarity tests for triangles:

1. If all three pairs of corresponding sides in a pair of triangles are in proportion, then the triangles are similar.
2. If two pairs of corresponding angles in a pair of triangles are congruent, then the triangles are similar.
3. If two pairs of corresponding sides in a pair of triangles are in proportion and the included angles are congruent, then the triangles are similar.

The areas of similar triangles are proportional to the squares of the corresponding linear parts (such as sides, altitudes, diagonals, etc.).

4°. The line connecting the midpoints of two sides of a triangle is called a *midline* of the triangle. The midline is parallel to and half as long as the third side (Fig. 3.1*b*).

Let a, b, and c be the lengths of the sides of a triangle; let α, β, and γ be the respective opposite angles (Fig. 3.1*a*); let R and r be the circumradius and the inradius, respectively; and let $p = \frac{1}{2}(a+b+c)$ be the semiperimeter.

Table 3.1 represents the basic properties and relations characterizing triangles.

TABLE 3.1
Basic properties and relations characterizing plane triangles

No.	The name of property	Properties and relations
1	Triangle inequality	The length of any side of a triangle does not exceed the sum of lengths of the other two sides
2	Sum of angles of a triangle	$\alpha+\beta+\gamma=180^\circ$
3	Law of sines	$\dfrac{a}{\sin\alpha}=\dfrac{b}{\sin\beta}=\dfrac{c}{\sin\gamma}=2R$
4	Law of cosines	$c^2=a^2+b^2-2ab\cos\gamma$
5	Law of tangents	$\dfrac{a+b}{a-b}=\dfrac{\tan\left[\frac{1}{2}(\alpha+\beta)\right]}{\tan\left[\frac{1}{2}(\alpha-\beta)\right]}=\dfrac{\cot\left(\frac{1}{2}\gamma\right)}{\tan\left[\frac{1}{2}(\alpha-\beta)\right]}$
6	Theorem on projections (law of cosines)	$c=a\cos\beta+b\cos\alpha$
7	Trigonometric angle formulas	$\sin\dfrac{\gamma}{2}=\sqrt{\dfrac{(p-a)(p-b)}{ab}},\quad \cos\dfrac{\gamma}{2}=\sqrt{\dfrac{p(p-c)}{ab}},$ $\tan\dfrac{\gamma}{2}=\sqrt{\dfrac{(p-a)(p-b)}{p(p-c)}},\quad \sin\gamma=\dfrac{2}{ab}\sqrt{p(p-a)(p-b)(p-c)}$
8	Law of tangents	$\tan\gamma=\dfrac{c\sin\alpha}{b-c\cos\alpha}=\dfrac{c\sin\beta}{a-c\cos\beta}$
9	Mollweide's formulas	$\dfrac{a+b}{c}=\dfrac{\cos\left[\frac{1}{2}(\alpha-\beta)\right]}{\sin\left(\frac{1}{2}\gamma\right)}=\dfrac{\cos\left[\frac{1}{2}(\alpha-\beta)\right]}{\cos\left[\frac{1}{2}(\alpha+\beta)\right]},$ $\dfrac{a-b}{c}=\dfrac{\sin\left[\frac{1}{2}(\alpha-\beta)\right]}{\cos\left(\frac{1}{2}\gamma\right)}=\dfrac{\sin\left[\frac{1}{2}(\alpha-\beta)\right]}{\sin\left[\frac{1}{2}(\alpha+\beta)\right]}$

Table 3.2 permits one to find the sides and angles of an arbitrary triangle if three appropriately chosen sides and/or angles are given. From the relations given in Tables 3.1 and 3.2, one can derive all missing relations by cyclic permutations of the sides a, b, and c and the angles α, β, and γ.

TABLE 3.2
Solution of plane triangles

No.	Three parts specified	Formulas for the remaining parts
1	Three sides a, b, c	*First method.* One of the angles is determined by the law of cosines, $\cos\alpha = \dfrac{b^2+c^2-a^2}{2bc}$. Then either the law of sines or the law of cosines is applied. *Second method.* One of the angles is determined by trigonometric angle formulas. Further proceed in a similar way. **Remark.** The sum of lengths of any two sides must be greater than the length of the third side.
2	Two sides a, b and the included angle γ	*First method.* The side c is determined by the law of cosines, $c = \sqrt{a^2+b^2-2ab\cos\gamma}$. The angle α is determined by either the law of cosines or the law of sines. The angle β is determined from the sum of angles in triangle, $\beta = 180^\circ - \alpha - \gamma$. *Second method.* $\alpha+\beta$ is found from the sum of angles in triangle, $\alpha+\beta = 180^\circ - \gamma$; $\alpha-\beta$ is found from the law of tangents, $\tan\dfrac{\alpha-\beta}{2} = \dfrac{a-b}{a+b}\cot\dfrac{\gamma}{2}$. Then α and β can be found. The third side c is determined by either the law of cosines or the law of sines.
3	A side c and the two angles α, β adjacent to it	The third angle γ is found from the sum of angles in triangle, $\gamma = 180^\circ - \alpha - \beta$. Sides a and b are determined by the law of sines.
4	Two sides a, b and the angle α opposite one of them	The second angle is determined by the law of sines, $\sin\beta = \dfrac{b}{a}\sin\alpha$. The third angle is $\gamma = 180^\circ - \alpha - \beta$. The third side is determined by the law of sines, $c = a\dfrac{\sin\gamma}{\sin\alpha}$. **Remark.** Five cases are possible: 1. $a > b$; i.e., the angle is opposite the greater side. Then $\alpha > \beta$, $\beta < 90^\circ$ (the larger angle is opposite the larger side), and the triangle is determined uniquely. 2. $a = b$; i.e., the triangle is isosceles and is determined uniquely. 3. $a < b$ and $b\sin\alpha < a$. Then there are two solutions, $\beta_1 + \beta_2 = 180^\circ$. 4. $a < b$ and $b\sin\alpha = a$. Then the solution is unique, $\beta = 90^\circ$. 5. $a < b$ and $b\sin\alpha > a$. Then there are no solutions.

3.1.1-2. Medians, angle bisectors, and altitudes of triangle.

A straight line through a vertex of a triangle and the midpoint of the opposite side is called a *median* of the triangle (Fig. 3.2*a*). The three medians of a triangle intersect in a single point lying strictly inside the triangle, which is called the *centroid* or *center of gravity* of the triangle. This point cuts the medians in the ratio 2:1 (counting from the corresponding vertices).

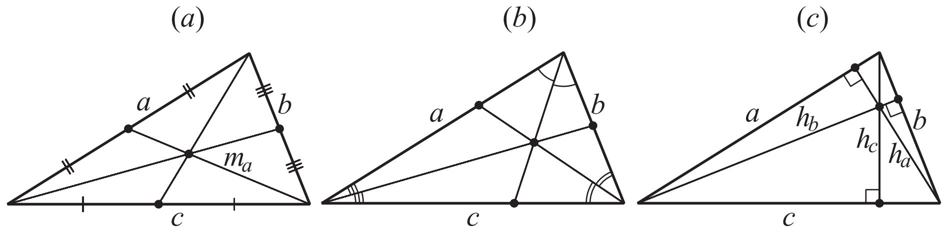

Figure 3.2. Medians (a), angle bisectors (b), and altitudes (c) of a triangle.

The length of the median m_a to the side a opposite the angle α is equal to

$$m_a = \frac{1}{2}\sqrt{2(b^2+c^2)-a^2} = \frac{1}{2}\sqrt{a^2+b^2+2ab\cos\gamma}. \qquad (3.1.1.1)$$

An *angle bisector* of a triangle is a line segment between a vertex and a point of the opposite side and dividing the angle at that vertex into two equal parts (Fig. 3.2*b*). The three angle bisectors intersect in a single point lying strictly inside the triangle. This point is equidistant from all sides and is called the *incenter* (the center of the *incircle* of the triangle). Concerning the radius r of the incircle, see Paragraph 3.1.1-3. The angle bisector through a vertex cuts the opposite side in ratio proportional to the adjacent sides of the triangle.

The length of the angle bisector l_a drawn to the side a is given by the formulas

$$\begin{aligned} l_a &= \sqrt{bc-b_1c_1} = \frac{\sqrt{bc[(b+c)^2-a^2]}}{b+c} = \frac{\sqrt{4p(p-a)bc}}{b+c}, \\ l_a &= \frac{2cb\cos\left(\frac{1}{2}\alpha\right)}{b+c} = 2R\frac{\sin\beta\sin\gamma}{\cos\left[\frac{1}{2}(\beta-\gamma)\right]} = 2p\frac{\sin\left(\frac{1}{2}\beta\right)\sin\left(\frac{1}{2}\gamma\right)}{\sin\beta+\sin\gamma}, \end{aligned} \qquad (3.1.1.2)$$

where b_1 and c_1 are the segments of the side a cut by bisector l_a and adjacent to the sides b and c, respectively, and R is the circumradius (see Paragraph 3.1.1-3).

An *altitude* of a triangle is a straight line passing through a vertex and perpendicular to the straight line containing the opposite side (Fig. 3.2*c*). The three altitudes of a triangle intersect in a single point, called the *orthocenter* of the triangle.

The length of the altitude h_a to the side a is given by the formulas

$$\begin{aligned} h_a &= b\sin\gamma = c\sin\beta = \frac{bc}{2R}, \\ h_a &= 2(p-a)\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2} = 2(p-b)\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\cos\frac{\gamma}{2}. \end{aligned} \qquad (3.1.1.3)$$

The lengths of the altitude, the angle bisector, and the median through the same vertex satisfy the inequality $h_a \le l_a \le m_a$. If $h_a = l_a = m_a$, then the triangle is isosceles; moreover, the first equality implies the second, and vice versa.

3.1.1-3. Circumcircle, incircle, and excircles.

A straight line passing through the midpoint of a segment and perpendicular to it is called the *perpendicular bisector* of the segment. The circle passing through the vertices of a triangle is called the *circumcircle* of the triangle. The center O_1 of the circumcircle, called the *circumcenter*, is the point where the perpendicular bisectors of the sides of the triangle

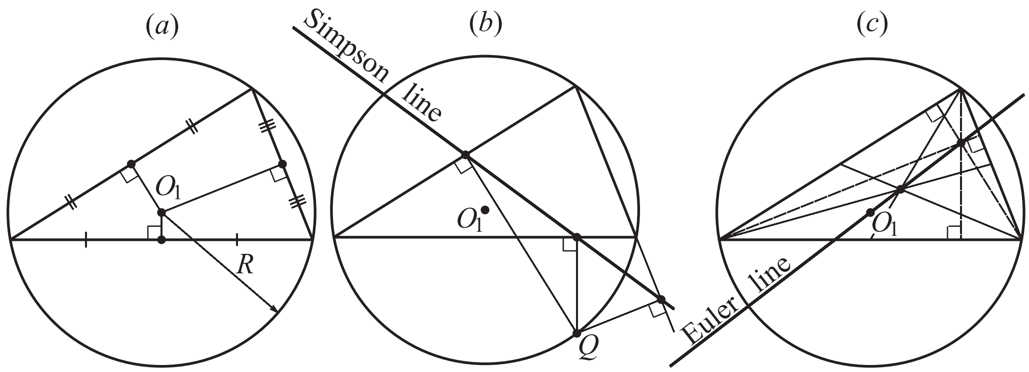

Figure 3.3. The circumcircle of a triangle. The circumcenter (*a*), the Simpson line (*b*), and the Euler line (*c*).

meet (Fig. 3.3*a*). The feet of the perpendiculars drawn from a point Q on the circumcircle to the three sides of the triangle lie on the same straight line called the *Simpson line* of Q with respect to the triangle (Fig. 3.3*b*). The circumcenter, the orthocenter, and the centroid lie on a single line, called the *Euler line* (Fig. 3.3*c*).

The circle tangent to the three sides of a triangle and lying inside the triangle is called the *incircle* of the triangle. The center O_2 of the incircle (the *incenter*) is the point where the angle bisectors meet (Fig. 3.4*a*). The straight lines connecting the vertices of a triangle with the points at which the incircle is tangent to the respective opposite sides intersect in a single point G called the *Gergonne point* (Fig. 3.4*b*).

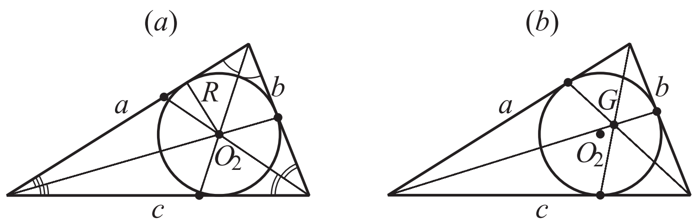

Figure 3.4. The incircle of a triangle (*a*). The incenter and the Gergonne point (*b*).

The circle tangent to one side of a triangle and to the extensions of the other two sides is called an *excircle* of the triangle. Each triangle has three excircles. The center of an excircle (an *excenter*) is the point of concurrency of two external angle bisectors and an interior angle bisector. The straight lines connecting the vertices of a triangle with the points at which the respective opposite sides are tangent to the excircles intersect in a single point N, called the *Nagel point* (Fig. 3.5).

The inradius r, the circumradius R, and the exradii ρ_a, ρ_b, and ρ_c satisfy the relations

$$r = \sqrt{\frac{(p-a)(p-b)(p-c)}{p}} = p\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}$$

$$= 4R\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2} = (p-c)\tan\frac{\gamma}{2} = \frac{S}{p}, \qquad (3.1.1.4)$$

$$R = \frac{a}{2\sin\alpha} = \frac{b}{2\sin\beta} = \frac{c}{2\sin\gamma} = \frac{abc}{4S} = \frac{p}{4\cos\left(\frac{1}{2}\alpha\right)\cos\left(\frac{1}{2}\beta\right)\cos\left(\frac{1}{2}\gamma\right)}, \qquad (3.1.1.5)$$

$$\frac{1}{r} = \frac{1}{\rho_a} + \frac{1}{\rho_b} + \frac{1}{\rho_c}. \qquad (3.1.1.6)$$

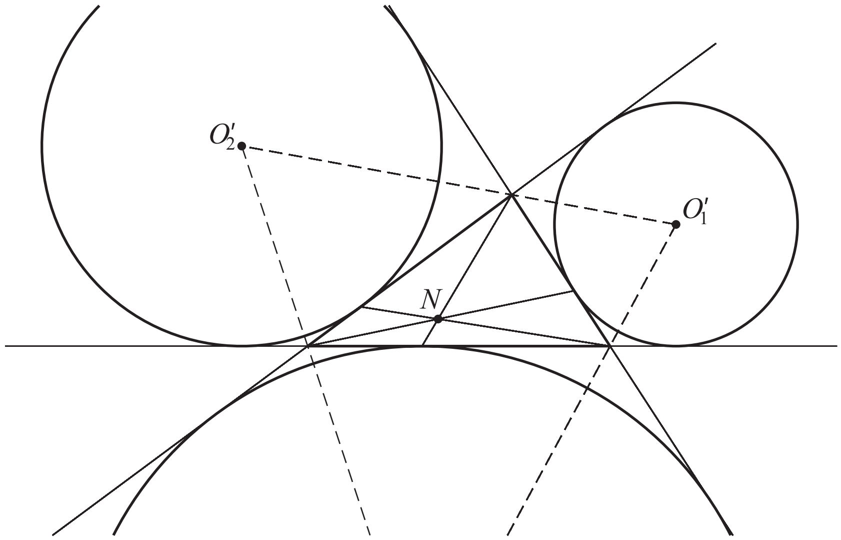

Figure 3.5. Excircles of a triangle. The Nagel point.

The distance d_1 between the circumcenter and the incenter and the distance d_2 between the circumcenter and the excenter are given by the expressions

$$d_1 = \sqrt{R^2 - 2Rr}, \tag{3.1.1.7}$$
$$d_2 = \sqrt{R^2 + 2R\rho_a}. \tag{3.1.1.8}$$

3.1.1-4. Area of a triangle.

The area S of a triangle is given by the formulas

$$\begin{aligned} S &= ah_a = \frac{1}{2}ab\sin\gamma = rp,\\ S &= \sqrt{p(p-a)(p-b)(p-c)} \quad (\textit{Heron's formula}),\\ S &= \frac{abc}{4R} = 2R^2\sin\alpha\sin\beta\sin\gamma,\\ S &= c^2\frac{\sin\alpha\sin\beta}{2\sin\gamma} = c^2\frac{\sin\alpha\sin\beta}{2\sin(\alpha+\beta)}. \end{aligned} \tag{3.1.1.9}$$

3.1.1-5. Theorems about points and lines related to triangle.

CEVA'S THEOREM. *Let points C_1, A_1, and B_1 lie on the sides AB, BC, and CA, respectively, of a triangle (Fig. 3.6). The straight lines AA_1, BB_1, and CC_1 are concurrent or parallel if and only if*

$$\frac{AC_1}{C_1B}\cdot\frac{BA_1}{A_1C}\cdot\frac{CB_1}{B_1A} = 1. \tag{3.1.1.10}$$

STEWART'S THEOREM. *If a straight line through a vertex of a triangle divides the opposite side into segments of lengths m and n (Fig. 3.7), then*

$$(m+n)(p^2+mn) = b^2m + c^2n. \tag{3.1.1.11}$$

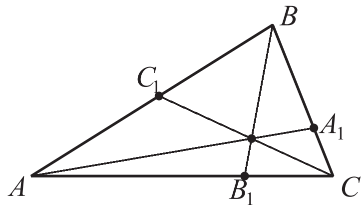

Figure 3.6. Ceva's theorem.

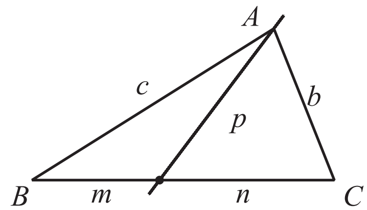

Figure 3.7. Stewart's theorem.

MENELAUS'S THEOREM. *If a straight line intersects sides AB, BC, and CA of a triangle (Fig. 3.8) or their extensions at points C_1, A_1, and B_1, respectively, then*

$$\frac{AC_1}{C_1B} \cdot \frac{BA_1}{A_1C} \cdot \frac{CB_1}{B_1A} = -1. \tag{3.1.1.12}$$

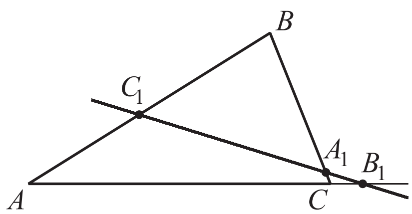

Figure 3.8. Menelaus's theorem.

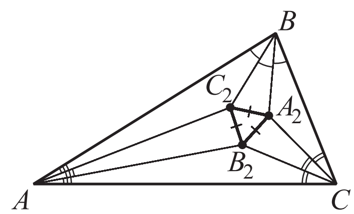

Figure 3.9. Morley's theorem.

Straight lines dividing the interior angles of a triangle into three equal parts are called *angle trisectors*.

MORLEY'S THEOREM. *The three points of intersection of adjacent angle trisectors of a triangle form an equilateral triangle (Fig. 3.9).*

In a triangle, the midpoints of the three sides, the feet of the three altitudes, and the midpoints of the segments of the altitudes between the orthocenter and the vertices all lie on a single circle, the *nine-point circle* (Fig. 3.10).

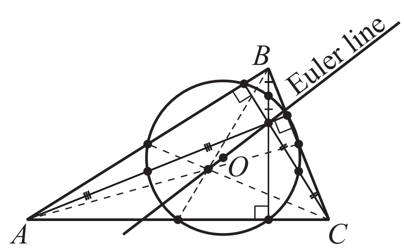

Figure 3.10. Nine-point circle.

FEUERBACH'S THEOREM. *The nine-point circle is tangent to the incircle and the three excircles. The points of tangency are called the Feuerbach points. The center of the nine-point circle lies on the Euler line (see Paragraph 3.1.1-3).*

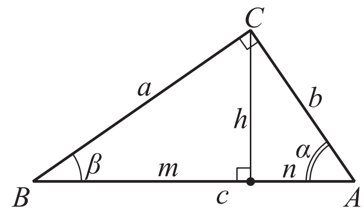

Figure 3.11. A right triangle.

3.1.1-6. Right (right-angled) triangles.

A *right* triangle is a triangle with a right angle. The side opposite the right angle is called the *hypotenuse*, and the other two sides are called the *legs* (Fig. 3.11).

The hypotenuse c, the legs a and b, and the angles α and β opposite the legs satisfy the following relations:

$$\begin{aligned} &\alpha + \beta = 90^\circ; \\ &\sin\alpha = \cos\beta = \frac{a}{c}, \quad \sin\beta = \cos\alpha = \frac{b}{c}, \\ &\tan\alpha = \cot\beta = \frac{a}{b}, \quad \tan\beta = \cot\alpha = \frac{b}{a}. \end{aligned} \tag{3.1.1.13}$$

One also has

$$a^2 + b^2 = c^2 \quad \text{(PYTHAGOREAN THEOREM)}, \tag{3.1.1.14}$$

$$h^2 = mn, \quad a^2 = mc, \quad b^2 = nc, \tag{3.1.1.15}$$

where h is the length of the altitude drawn to the hypotenuse; moreover, the altitude cuts the hypotenuse into segments of lengths m and n.

In a right triangle, the length of the median m_c drawn from the vertex of the right angle coincides with the circumradius R and is equal to half the length of the hypotenuse c, $m_c = R = \frac{1}{2}c$. The inradius is given by the formula $r = \frac{1}{2}(a + b - c)$. The area of the right triangle is $S = ah_a = \frac{1}{2}ab$ (see also Paragraphs 3.1.1-2 to 3.1.1-4).

3.1.1-7. Isosceles and equilateral triangles.

1°. An *isosceles* triangle is a triangle with two equal sides. These sides are called the *legs*, and the third side is called the *base* (Fig. 3.12*a*).

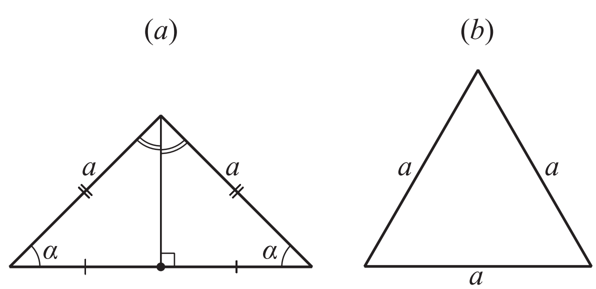

Figure 3.12. An isosceles triangle (*a*). An equilateral triangle (*b*).

Properties of isosceles triangles:

1. In an isosceles triangle, the angles adjacent to the base are equal.
2. In an isosceles triangle, the median drawn to the base is the angle bisector and the altitude.
3. In an isosceles triangle, the sum of distances from a point of the base to the legs is constant.

Criteria for a triangle to be isosceles:

1. If two angles in a triangle are equal, then the triangle is isosceles.
2. If a median in a triangle is also an altitude, then the triangle is isosceles.
3. If a bisector in a triangle is also an altitude, then the triangle is isosceles.

2°. An *equilateral* (or *regular*) triangle is a triangle with all three sides equal (Fig. 3.12*b*). All angles of an equilateral triangle are equal to 60°. In an equilateral triangle, the circumradius R and the inradius r satisfy the relation $R = 2r$.

For an equilateral triangle with side length a, the circumradius and the inradius are given by the formulas $R = \frac{\sqrt{3}}{3}a$ and $r = \frac{\sqrt{3}}{6}a$, and the area is equal to $S = \frac{\sqrt{3}}{4}a^2$.

3.1.2. Polygons

3.1.2-1. Polygons. Basic information.

A *polygon* is a plane figure bounded by a closed broken line, i.e., a line obtained if one takes n distinct points such that no three successive points are collinear and draws a straight line segment between each of these points and its successor as well as between the last point and the first point (Fig. 3.13*a*). The segments forming a polygon are called the *sides* (or *edges*), and the points themselves are called the *vertices* of the polygon. Two sides sharing a vertex, as well as two successive vertices (the endpoints of the same edge), are said to be adjacent. A polygon can be self-intersecting, but the points of self-intersection should not be vertices (Fig. 3.13*b*). A polygon is said to be *plane* if its vertices are coplanar. A polygon is said to be *simple* if its nonadjacent sides do not have common interior or endpoints. A polygon is said to be *convex* if it lies on one side of any straight line passing through two neighboring vertices (Fig. 3.13*c*). In what follows, we consider only plane simple convex polygons.

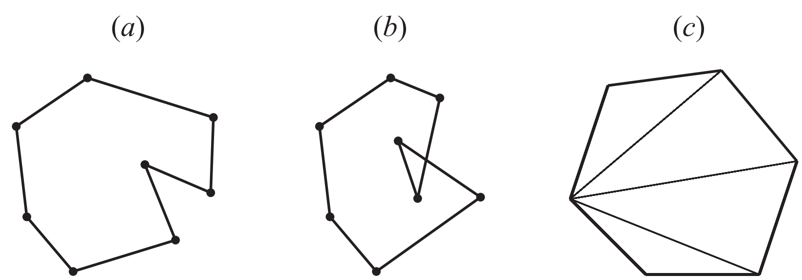

Figure 3.13. Polygons. Nonself-intersecting (*a*), self-intersecting (*b*), and convex (*c*) polygon.

An *(interior) angle* of a convex polygon is the angle between two sides meeting in a vertex. An angle adjacent to an interior angle is called an *external angle* of the convex polygon. A convex polygon is said to be *inscribed* in a circle if all of its vertices lie on the circle. A polygon is said to be *circumscribed* about a circle if all of its sides are tangent to the circle.

For a convex polygon with n sides, the sum of interior angles is equal to $180°(n-2)$, and the sum of external angles is equal to 360°.

One can find the area of an arbitrary polygon by dividing it into triangles.

3.1.2-2. Properties of quadrilaterals.

1. The diagonals of a convex quadrilateral meet.
2. The sum of interior angles of a convex quadrilateral is equal to 360° (Fig. 3.14*a* and *b*).
3. The lengths of the sides a, b, c, and d, the diagonals d_1 and d_2, and the segment m connecting the midpoints of the diagonals satisfy the relation $a^2 + b^2 + c^2 + d^2 = d_1^2 + d_2^2 + 4m^2$.
4. A convex quadrilateral is circumscribed if and only if $a + c = b + d$.
5. A convex quadrilateral is inscribed if and only if $\alpha + \gamma = \beta + \delta$.
6. The relation $ac + bd = d_1 d_2$ holds for inscribed quadrilaterals (PTOLEMY'S THEOREM).

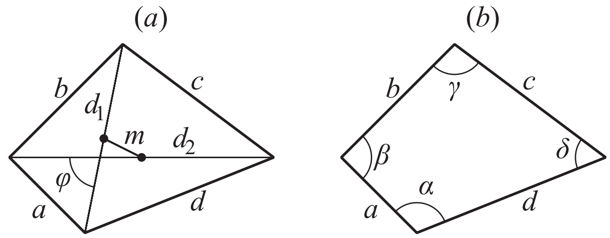

Figure 3.14. Quadrilaterals.

3.1.2-3. Areas of quadrilaterals.

The area of a convex quadrilateral is equal to

$$S = \frac{1}{2} d_1 d_2 \sin\varphi = \sqrt{p(p-a)(p-b)(p-c)(p-d) - abcd\cos^2\frac{\beta+\delta}{2}}, \qquad (3.1.2.1)$$

where φ is the angle between the diagonals d_1 and d_2 and $p = \frac{1}{2}(a + b + c + d)$.

The area of an inscribed quadrilateral is

$$S = \sqrt{p(p-a)(p-b)(p-c)(p-d)}. \qquad (3.1.2.2)$$

The area of a circumscribed quadrilateral is

$$S = \sqrt{abcd\sin^2\frac{\beta+\delta}{2}}. \qquad (3.1.2.3)$$

If a quadrilateral is simultaneously inscribed and circumscribed, then

$$S = \sqrt{abcd}. \qquad (3.1.2.4)$$

3.1.2-4. Basic quadrilaterals.

1°. A *parallelogram* is a quadrilateral such that both pairs of opposite sides are parallel (Fig. 3.15*a*).

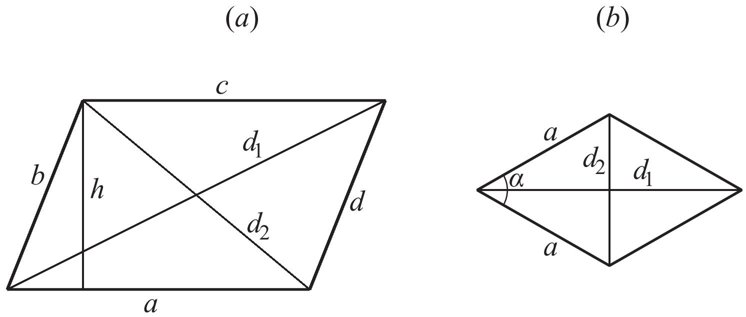

Figure 3.15. A parallelogram (*a*) and a rhombus (*b*).

Attributes of parallelograms (a quadrilateral is a parallelogram if):

1. Both pairs of opposite sides have equal length.
2. Both pairs of opposite angles are equal.
3. Two opposite sides are parallel and have equal length.
4. The diagonals meet and bisect each other.

Properties of parallelograms:

1. The diagonals meet and bisect each other.
2. Opposite sides have equal length, and opposite angles are equal.
3. The diagonals and the sides satisfy the relation $d_1^2 + d_2^2 = 2(a^2 + b^2)$.
4. The area of a parallelogram is $S = ah$, where h is the altitude (see also Paragraph 3.1.2-3).

2°. A *rhombus* is a parallelogram in which all sides are of equal length (Fig. 3.15*b*).

Properties of rhombi:

1. The diagonals are perpendicular.
2. The diagonals are angle bisectors.
3. The area of a rhombus is $S = ah = a^2 \sin\alpha = \frac{1}{2}d_1 d_2$.

3°. A *rectangle* is a parallelogram in which all angles are right angles (Fig. 3.16*a*).

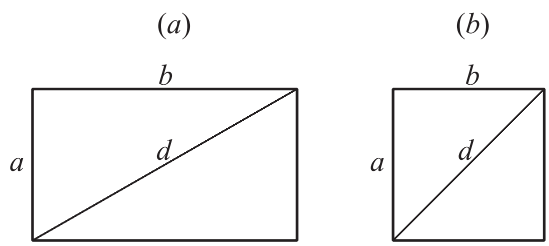

Figure 3.16. A rectangle (*a*) and a square (*b*).

Properties of rectangles:

1. The diagonals have equal lengths.
2. The area of a rectangle is $S = ab$.

4°. A *square* is a rectangle in which all sides have equal lengths (Fig. 3.16*b*). A square is also a special case of a rhombus (all angles are right angles).

Properties of squares:

1. All angles are right angles.
2. The diagonals are equal to $d = a\sqrt{2}$.
3. The diagonals meet at a right angle and are angle bisectors.
4. The area of a square is equal to $S = a^2 = \frac{1}{2}d^2$.

5°. A *trapezoid* is a quadrilateral in which two sides are parallel and the other two sides are nonparallel (Fig. 3.17). The parallel sides a and b are called the *bases* of the trapezoid, and the other two sides are called the *legs*. In an *isosceles* trapezoid, the legs are of equal length. The line segment connecting the midpoints of the legs is called the *median* of the trapezoid. The length of the median is equal to half the sum of the lengths of the bases, $m = \frac{1}{2}(a + b)$.

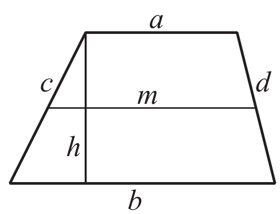

Figure 3.17. A trapezoid.

The perpendicular distance between the bases is called the *altitude* of a trapezoid.

Properties of trapezoids:

1. A trapezoid is circumscribed if and only if $a + b = c + d$.
2. A trapezoid is inscribed if and only if it is isosceles.
3. The area of a trapezoid is $S = \frac{1}{2}(a + b)h = mh = \frac{1}{2}d_1 d_2 \sin\varphi$, where φ is the angle between the diagonals d_1 and d_2.
4. The segment connecting the midpoints of the diagonals is parallel to the bases and has the length $\frac{1}{2}(b - a)$.

Example 1. Consider an application of plane geometry to measuring distances in geodesy. Suppose that the angles α, β, γ, and δ between a straight line AB and the directions to points D and C are known at points A and B (Fig. 3.18*a*). Suppose also that the distance $a = AB$ (or $b = DC$) is known and the task is to find the distance $b = DC$ (or $a = AB$).

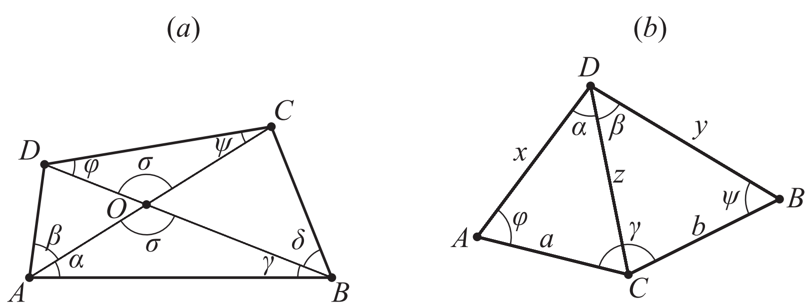

Figure 3.18. Applications of plane geometry in geodesy.

Let us find the angles φ and ψ. Since σ is the angle at the vertex O in both triangles AOB and DOC, it follows that $\alpha + \gamma = \varphi + \psi$. Let $\varepsilon_1 = \frac{1}{2}(\varphi + \psi)$. We twice apply the law of sines (Table 3.1) and find the half-difference of the desired angles. The main formulas read

$$\frac{AD}{a} = \frac{\sin\gamma}{\sin(\pi - \alpha - \beta - \gamma)} = \frac{\sin\gamma}{\sin(\alpha + \beta + \gamma)}, \qquad \frac{BC}{a} = \frac{\sin\alpha}{\sin(\alpha + \gamma + \delta)},$$

$$\frac{b}{AD} = \frac{\sin\beta}{\sin\gamma}, \quad \frac{b}{BC} = \frac{\sin\delta}{\sin\varphi}.$$

These relations imply that

$$\frac{b}{a} = \frac{\sin\beta\sin\gamma}{\sin\psi\sin(\alpha+\beta+\gamma)} = \frac{\sin\delta\sin\alpha}{\sin\varphi\sin(\alpha+\gamma+\delta)} \tag{3.1.2.5}$$

and hence

$$\frac{\sin\varphi}{\sin\psi} = \frac{\sin\delta\sin\alpha\sin(\alpha+\beta+\gamma)}{\sin\beta\sin\gamma\sin(\alpha+\gamma+\delta)} = \cot\eta,$$

where η is an auxiliary angle. By adding and subtracting, we obtain

$$\frac{\sin\varphi - \sin\psi}{\sin\varphi + \sin\psi} = \frac{\cot\eta - 1}{\cot\eta + 1}, \quad \frac{2\cos\left[\frac{1}{2}(\varphi+\psi)\right]\sin\left[\frac{1}{2}(\varphi-\psi)\right]}{2\sin\left[\frac{1}{2}(\varphi+\psi)\right]\cos\left[\frac{1}{2}(\varphi-\psi)\right]} = \frac{\cot\left(\frac{1}{4}\pi\right)\cot\eta - 1}{\cot\eta + \cot\left(\frac{1}{4}\pi\right)},$$

$$\tan\frac{\varphi-\psi}{2} = \tan\frac{\varphi+\psi}{2}\cot\left(\frac{\pi}{4}+\eta\right) = \tan\frac{\alpha+\gamma}{2}\cot\left(\frac{\pi}{4}+\eta\right).$$

From this we find $\varepsilon_2 = \frac{1}{2}(\varphi - \psi)$ and, substituting $\varphi = \varepsilon_1 + \varepsilon_2$ and $\psi = \varepsilon_1 - \varepsilon_2$ into (3.1.2.5), obtain the desired distance.

Example 2. Suppose that the mutual position of three points A, B, and C is determined by the segments $AC = a$ and $BC = b$, and the angle $\angle ACB = \gamma$. Suppose that the following angles have been measured at some point D: $\angle CDA = \alpha$ and $\angle CDB = \beta$.

In the general case, one can find the position of point D with respect to A, B, and C, i.e., uniquely determine the segments x, y, and z (Fig. 3.18*b*). For this to be possible, it is necessary that D does not lie on the circumcircle of the triangle ABC. We have

$$\varphi + \psi = 2\pi - (\alpha+\beta+\gamma) = 2\varepsilon_1. \tag{3.1.2.6}$$

By the law of sines (Table 3.1), we obtain

$$\sin\varphi = \frac{z}{a}\sin\alpha, \quad \sin\psi = \frac{z}{b}\sin\beta, \tag{3.1.2.7}$$

which implies that

$$\frac{\sin\varphi}{\sin\psi} = \frac{b\sin\alpha}{a\sin\beta} = \cot\eta, \tag{3.1.2.8}$$

where η is an auxiliary angle. We find the angles φ and ψ from (3.1.2.6) and (3.1.2.8), substitute them into (3.1.2.7) to determine z, and finally apply the law of sines to obtain x and y.

3.1.2-5. Regular polygons.

A convex polygon is said to be *regular* if all of its sides have the same length and all of its interior angles are equal. A convex n-gon is regular if and only if it is taken to itself by the rotation by an angle of $2\pi/n$ about some point O. The point O is called the *center* of the regular polygon. The angle between two rays issuing from the center and passing through two neighboring vertices is called the *central angle* (Fig. 3.19).

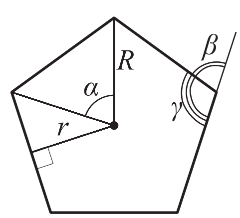

Figure 3.19. A regular polygon.

Properties of regular polygons:

1. The center is equidistant from all vertices as well as from all sides of a regular polygon.
2. A regular polygon is simultaneously inscribed and circumscribed; the centers of the circumcircle and the incircle coincide with the center of the polygon itself.
3. In a regular polygon, the central angle is $\alpha = 360°/n$, the external angle is $\beta = 360°/n$, and the interior angle is $\gamma = 180° - \beta$.
4. The circumradius R, the inradius r, and the side length a of a regular polygon satisfy the relations
$$a = 2\sqrt{R^2 - r^2} = 2R\sin\frac{\alpha}{2} = 2r\tan\frac{\alpha}{2}. \qquad (3.1.2.9)$$
5. The area S of a regular n-gon is given by the formula
$$S = \frac{arn}{2} = nr^2\tan\frac{\alpha}{2} = nR^2\sin\frac{\alpha}{2} = \frac{1}{4}na^2\cot\frac{\alpha}{2}. \qquad (3.1.2.10)$$

Table 3.3 presents several useful formulas for regular polygons.

TABLE 3.3
Regular polygons (a is the side length)

No.	Name	Inradius r	Circumradius R	Area S
1	Regular polygon	$\dfrac{a}{2\tan\frac{\pi}{n}}$	$\dfrac{a}{2\sin\frac{\pi}{n}}$	$\frac{1}{2}arn$
2	Triangle	$\frac{\sqrt{3}}{6}a$	$\frac{\sqrt{3}}{3}a$	$\frac{\sqrt{3}}{4}a^2$
3	Square	$\frac{1}{2}a$	$\frac{1}{\sqrt{2}}a$	a^2
4	Pentagon	$\sqrt{\frac{5+2\sqrt{5}}{20}}a$	$\sqrt{\frac{5+\sqrt{5}}{10}}a$	$\frac{\sqrt{25+10\sqrt{5}}}{4}a^2$
5	Hexagon	$\frac{\sqrt{3}}{2}a$	a	$\frac{3\sqrt{3}}{2}a^2$
6	Octagon	$\frac{1+\sqrt{2}}{2}a$	$\frac{\sqrt{2+\sqrt{2}}}{2}a$	$2(1+\sqrt{2})a^2$
7	Enneagon	$\frac{5+2\sqrt{5}}{2}a$	$\frac{1+\sqrt{5}}{2}a$	$\frac{\sqrt{5+2\sqrt{5}}}{2}a^2$
8	Dodecagon	$\frac{2+\sqrt{3}}{2}a$	$\frac{3+\sqrt{3}}{\sqrt{6}}a$	$3(2+\sqrt{3})a^2$

3.1.3. Circle

3.1.3-1. Some definitions and formulas.

The *circle of radius R centered at O* is the set of all points of the plane at a fixed distance R from a fixed point O (Fig. 3.20*a*). A plane figure bounded by a circle is called a *disk*. A segment connecting two points on a circle is called a *chord*. A chord passing through the center of a circle is called a *diameter* of the circle (Fig. 3.20*b*). The diameter length is $d = 2R$. A straight line that meets a circle at a single point is called a *tangent*, and the common point is called the *point of tangency* (Fig. 3.20*c*). An angle formed by two radii is called a *central angle*. An angle formed by two chords with a common endpoint is called an *inscribed angle*.

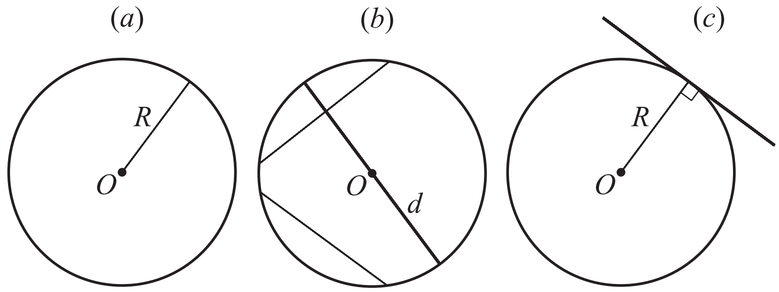

Figure 3.20. A circle (a). A diameter (b) and a tangent (c) of a circle.

Properties of circles and disks:

1. The circumference is $L = 2\pi R = \pi d = 2\sqrt{\pi S}$.
2. The area of a disk is $S = \pi R^2 = \frac{1}{4}\pi d^2 = \frac{1}{4}Ld$.
3. The diameter of a circle is a longest chord.
4. The diameter passing through the midpoint of the chord is perpendicular to the chord.
5. The radius drawn to the point of tangency is perpendicular to the tangent.
6. An inscribed angle is half the central angle subtended by the same chord, $\alpha = \frac{1}{2}\angle BOC$ (Fig. 3.21a).
7. The angle between a chord and the tangent to the circle at an endpoint of the chord is $\beta = \frac{1}{2}\angle AOC$ (Fig. 3.21a).
8. The angle between two chords is $\gamma = \frac{1}{2}(\breve{BC} + \breve{ED})$ (Fig. 3.21b).
9. The angle between two secants is $\alpha = \frac{1}{2}(\breve{DE} - \breve{BC})$ (Fig. 3.21c).

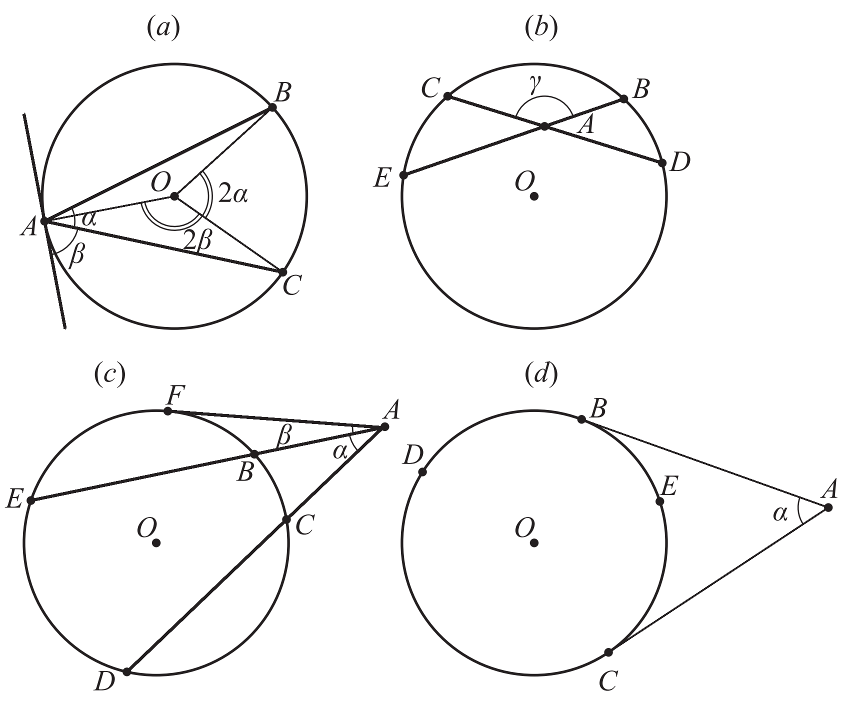

Figure 3.21. Properties of circles and disks.

10. The angle between a secant and the tangent to the circle at an endpoint of the secant is

$\beta = \frac{1}{2}(\breve{FE} - \breve{BF})$ (Fig. 3.21c).

11. The angle between two tangents is $\alpha = \frac{1}{2}(B\breve{DC} - B\breve{EC})$ (Fig. 3.21d).
12. If two chords meet, then $AC \cdot AD = AB \cdot AE = R^2 - m^2$ (Fig. 3.21b).
13. For secants, $AC \cdot AD = AB \cdot AE = m^2 - R^2$ (Fig. 3.21c).
14. For a tangent and a secant, $AF \cdot AF = AC \cdot AD$ (Fig. 3.21c).

3.1.3-2. Segment and sector.

A plane figure bounded by two radii and one of the subtending arcs is called a *(circular) sector*. A plane figure bounded by an arc and the corresponding chord is called a *segment* (Fig. 3.22a). If R is the radius of the circle, l is the arc length, a is the chord length, α is the central angle (in degrees), and h is the height of the segment, then the following formulas hold:

$$\begin{aligned} a &= 2\sqrt{2hR - h^2} = 2R \sin\frac{\alpha}{2}, \\ h &= R - \sqrt{R^2 - \frac{a^2}{4}} = R\left(1 - \cos\frac{\alpha}{2}\right) = \frac{a}{2}\tan\frac{\alpha}{4}, \\ l &= \frac{2\pi R\alpha}{360} \approx 0.01745\, R\alpha. \end{aligned} \tag{3.1.3.1}$$

The area of a circular sector is given by the formula

$$S = \frac{lR}{2} = \frac{\pi R^2 \alpha}{360} \approx 0.00873\, R^2 \alpha, \tag{3.1.3.2}$$

and the area of a segment not equal to a half-disk is given by the expression

$$S_1 = \frac{\pi R^2 \alpha}{360} \pm S_\Delta, \tag{3.1.3.3}$$

where S_Δ is the area of the triangle with vertices at the center of the disk and at the endpoints of the radii bounding the corresponding sector. One takes the minus sign for $\alpha < 180$ and the plus sign for $\alpha > 180$.

The arc length and the area of a segment can be found by the approximate formulas

$$\begin{aligned} l &\approx \frac{8b - a}{3}, \qquad l \approx \sqrt{a^2 + \frac{16h^2}{3}}, \\ S_1 &\approx \frac{h(6a + 8b)}{15}, \end{aligned} \tag{3.1.3.4}$$

where b is the chord of the half-segment (see Fig. 3.22a).

3.1.3-3. Annulus.

An *annulus* is a plane figure bounded by two concentric circles of distinct radii (Fig. 3.22b). Let R be the outer radius of an annulus (the radius of the outer bounding circle), and let r

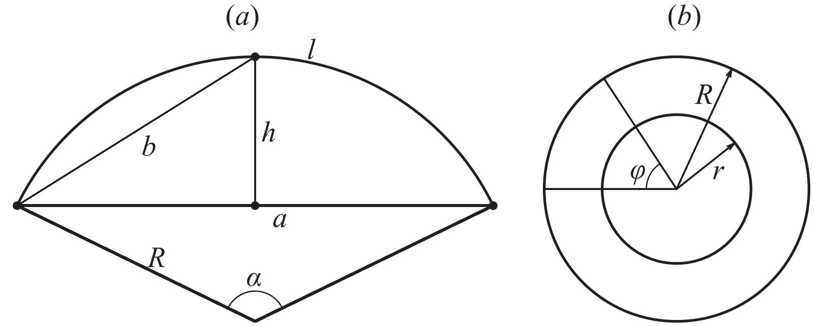

Figure 3.22. A segment (a) and an annulus (b).

be the inner radius (the radius of the inner bounding circle). Then the area of the annulus is given by the formula

$$S = \pi(R^2 - r^2) = \frac{\pi}{4}(D^2 - d^2) = 2\pi\rho\delta, \tag{3.1.3.5}$$

where $D = 2R$ and $d = 2r$ are the outer and inner diameters, $\rho = \frac{1}{2}(R + r)$ is the midradius, and $\delta = R - r$ is the width of the annulus.

The area of the part of the annulus contained in a sector of central angle φ, given in degrees (see Fig. 3.22b), is given by the formula

$$S = \frac{\pi\varphi}{360}(R^2 - r^2) = \frac{\pi\varphi}{1440}(D^2 - d^2) = \frac{\pi\varphi}{180}\rho\delta. \tag{3.1.3.6}$$

3.2. Solid Geometry

3.2.1. Straight Lines, Planes, and Angles in Space

3.2.1-1. Mutual arrangement of straight lines and planes.

1°. Two distinct straight lines lying in a single plane either have exactly one point of intersection or do not meet at all. In the latter case, they are said to be *parallel*. If two straight lines do not lie in a single plane, then they are called *skew lines*.

The angle between skew lines is determined as the angle between lines parallel to them and lying in a single plane (Fig. 3.23a). The distance between skew lines is the length of the straight line segment that meets both lines and is perpendicular to them.

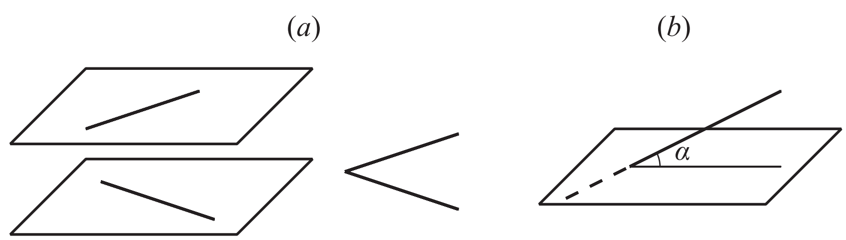

Figure 3.23. The angle between skew lines (a). The angle between a line and a plane (b).

2°. Two distinct planes either intersect in a straight line or do not have common points. In the latter case, they are said to be *parallel*. Coinciding planes are also assumed to be parallel. If two planes are perpendicular to a single straight line or each of them contains a pair of intersecting straight lines parallel to the corresponding lines in the other pair, then the planes are parallel.

3°. A straight line either entirely lies in the plane, meets the plane at a single point, or has no common points with the plane. In the last case, the line is said to be *parallel* to the plane.

The angle between a straight line and a plane is equal to the angle between the line and its projection onto the plane (Fig. 3.23*b*). If a straight line is perpendicular to two intersecting straight lines on a plane, then it is perpendicular to each line on the plane, i.e., *perpendicular to the plane*.

3.2.1-2. Polyhedral angles.

1°. A *dihedral angle* is a figure in space formed by two half-planes issuing from a single straight line as well as the part of space bounded by these half-planes. The half-planes are called the *faces* of the dihedral angle, and their common straight line is called the *edge*. A dihedral angle is measured by its linear angle ABC (Fig. 3.24*a*), i.e., by the angle between the perpendiculars raised to the edge DE of the dihedral angle in both planes (*faces*) at the same point.

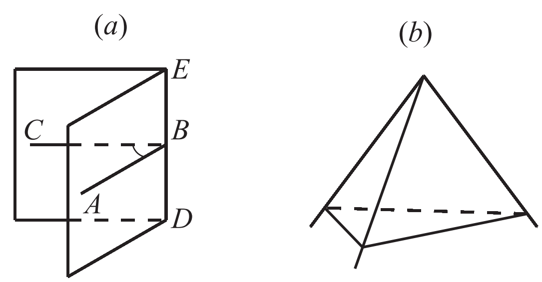

Figure 3.24. A dihedral (*a*) and a trihedral (*b*) angle.

2°. A part of space bounded by an infinite triangular pyramid is called a *trihedral angle* (Fig. 3.24*b*). The faces of this pyramid are called the *faces* of the trihedral angle, and the vertex of the pyramid is called the *vertex of a trihedral angle*. The rays in which the faces intersect are called the *edges* of a trihedral angle. The edges form *face angles*, and the faces form the dihedral angles of the trihedral angle. As a rule, one considers trihedral angles with dihedral angles less than π (or 180°), i.e., convex trihedral angles. Each face angle of a convex trihedral angle is less than the sum of the other two face angles and greater than their difference.

Two trihedral angles are equal if one of the following conditions is satisfied:

1. Two face angles, together with the included dihedral angle, of the first trihedral angle are equal to the respective parts (arranged in the same order) of the second trihedral angle.
2. Two dihedral angles, together with the included face angle, of the first trihedral angle are equal to the respective parts (arranged in the same order) of the second trihedral angle.
3. The three face angles of the first trihedral angle are equal to the respective face angles (arranged in the same order) of the second trihedral angle.
4. The three dihedral angles of the first trihedral angle are equal to the respective dihedral angles (arranged in the same order) of the second trihedral angle.

3°. A *polyhedral angle* $OABCDE$ (Fig. 3.25*a*) is formed by several planes (*faces*) having a single common point (the *vertex*) and successively intersecting along straight lines OA,

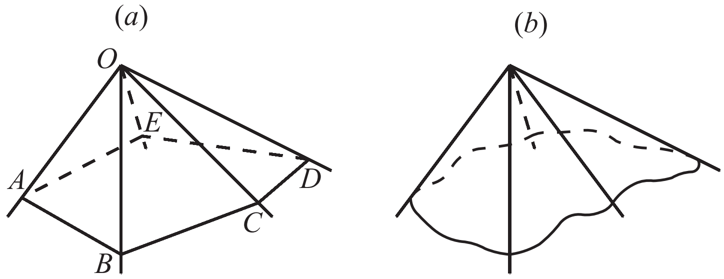

Figure 3.25. A polyhedral (*a*) and a solid (*b*) angle.

$OB, \ldots, OE$ (the *edges*). Two edges belonging to the same face form a *face angle* of the polyhedral angle, and two neighboring faces form a *dihedral angle*.

Polyhedral angles are equal (*congruent*) if one can be transformed into the other by translations and rotations. For polyhedral angles to be congruent, the corresponding parts (face and dihedral angles) must be equal. However, if the corresponding equal parts are arranged in reverse order, then the polyhedral angles cannot be transformed into each other by translations and rotations. In this case, they are said to be *symmetric*.

A convex polyhedral angle lies entirely on one side of each of its faces. The sum $\angle AOB + \angle BOC + \cdots + \angle EOA$ of face angles (Fig. 3.25*a*) of any convex polyhedral angle is less that 2π (or 360°).

4°. A *solid angle* is a part of space bounded by straight lines issuing from a single point (vertex) to all points of some closed curve (Fig. 3.25*b*). Trihedral and polyhedral angles are special cases of solid angles. A solid angle is measured by the area cut by the solid angle on the sphere of unit radius centered at the vertex. Solid angles are measured in steradians. The entire sphere forms a solid angle of 4π steradians.

3.2.2. Polyhedra

3.2.2-1. General concepts.

A *polyhedron* is a solid bounded by planes. In other words, a polyhedron is a set of finitely many plane polygons satisfying the following conditions:

1. Each side of each polygon is simultaneously a side of a unique other polygon, which is said to be adjacent to the first polygon (via this side).
2. From each of the polygons forming a polyhedron, one can reach any other polygon by successively passing to adjacent polygons.

These polygons are called the *faces*, their sides are called the *edges*, and their vertices are called the *vertices* of a polyhedron.

A polyhedron is said to be *convex* if it lies entirely on one side of the plane of any of its faces; if a polyhedron is convex, then so are its faces.

EULER'S THEOREM. *If the number of vertices in a convex polyhedron is e, the number of edges is f, and the number of faces is g, then $e + f - g = 2$.*

3.2.2-2. Prism. Parallelepiped.

1°. A *prism* is a polyhedron in which two faces are n-gons (the *base faces* of the prism) and the remaining n faces (*joining faces*) are parallelograms. The base faces of a prism are (*a*) (*b*)

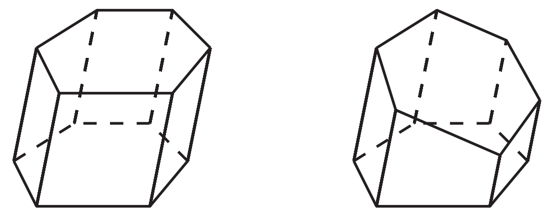

Figure 3.26. A prism (*a*) and a truncated prism (*b*).

equal (congruent) and lie in parallel planes (Fig. 3.26*a*). A *right* prism is a prism in which the joining faces are perpendicular to the base faces. A right prism is said to be *regular* if its base face is a regular polygon.

If l is the joining edge length, S is the area of the base face, H is the altitude of the prism, P_{sec} is the perimeter of a perpendicular section, and S_{sec} is the area of the perpendicular section, then the area of the lateral surface S_{lat} and the volume V of the prism can be determined by the formulas

$$\begin{aligned} S_{\text{lat}} &= P_{\text{sec}}\, l \\ V &= SH = S_{\text{sec}}\, l. \end{aligned} \tag{3.2.2.1}$$

The portion of a prism cut by a plane nonparallel to the base face is called a *truncated prism* (Fig. 3.26*b*). The volume of a truncated prism is

$$V = LP_1, \tag{3.2.2.2}$$

where L is the length of the segment connecting the centers of the base faces and P_1 is the area of the section of the prism by a plane perpendicular to this segment.

2°. A prism whose bases are parallelograms is called a *parallelepiped*. All four diagonals in a parallelepiped intersect at a single point and bisect each other (Fig. 3.27*a*). A parallelepiped is said to be *rectangular* if it is a right prism and its base faces are rectangles. In a rectangular parallelepiped, all diagonals are equal (Fig. 3.27*b*).

(*a*) (*b*)

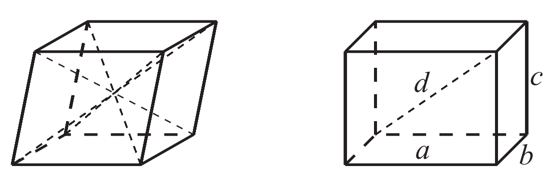

Figure 3.27. A parallelepiped (*a*) and a rectangular parallelepiped (*b*).

If a, b, and c are the lengths of the edges of a rectangular parallelepiped, then the diagonal d can be determined by the formula $d^2 = a^2 + b^2 + c^2$. The volume of a rectangular parallelepiped is given by the formula $V = abc$, and the lateral surface area is $S_{\text{lat}} = PH$, where P is the perimeter of the base face.

3°. A rectangular parallelepiped all of whose edges are equal ($a = b = c$) is called a *cube*. The diagonal of a cube is given by the formula $d^2 = 3a^2$. The volume of the cube is $V = a^3$, and the lateral surface area is $S_{lat} = 4a^2$.

3.2.2-3. Pyramid, obelisk, and wedge.

1°. A *pyramid* is a polyhedron in which one face (the *base* of the pyramid) is an arbitrary polygon and the other (*lateral*) faces are triangles with a common vertex, called the *apex* of the pyramid (Fig. 3.28*a*). The base of an n*-sided* pyramid is an n-gon. The perpendicular through the apex to the base of a pyramid is called the *altitude* of the pyramid.

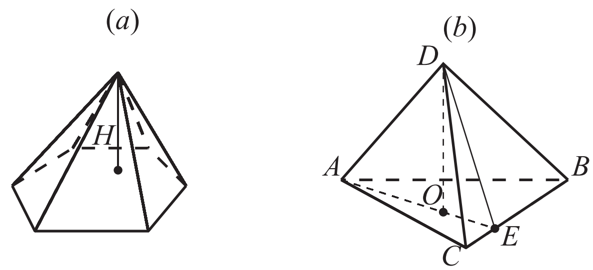

Figure 3.28. A pyramid (a). The attitude DO, the plane DAE, and the side BC in a triangular pyramid (b).

The volume of a pyramid is given by the formula

$$V = \frac{1}{3}SH, \tag{3.2.2.3}$$

where S is the area of the base and H is the altitude of the pyramid.

The apex of a pyramid is projected onto the circumcenter of the base if one of the following conditions is satisfied:

1. The lengths of all lateral edges are equal.
2. All lateral edges make equal angles with the base plane.

The apex of a pyramid is projected onto the incenter of the base if one of the following conditions is satisfied:

3. All lateral faces have equal apothems.
4. The angles between all lateral faces and the base are the same.

If DO is the altitude of the pyramid $ABCD$ and $DA \perp BC$, then the plane DAE is perpendicular to BC (Fig. 3.28*b*).

If the pyramid is cut by a plane (Fig. 3.29*a*) parallel to the base, then

$$\begin{gathered} \frac{SA_1}{A_1A} = \frac{SB_1}{B_1B} = \cdots = \frac{SO_1}{O_1O}, \\ \frac{S_{ABCDEF}}{S_{A_1B_1C_1D_1E_1F_1}} = \left(\frac{SO}{SO_1}\right)^2, \end{gathered} \tag{3.2.2.4}$$

where SO is the altitude of the pyramid, i.e., the segment of the perpendicular through the vertex to the base.

The altitude of a *triangular pyramid* passes through the orthocenter of the base if and only if all pairs of opposite edges of the pyramid are perpendicular. The volume of a triangular pyramid (Fig. 3.29*b*), where $DA = a$, $DB = b$, $DC = c$, $BC = p$, $AC = q$, and $AB = r$, is given by the formula

$$V^2 = \frac{1}{288} \begin{vmatrix} 0 & r^2 & q^2 & a^2 & 1 \\ r^2 & 0 & p^2 & b^2 & 1 \\ q^2 & p^2 & 0 & c^2 & 1 \\ a^2 & b^2 & c^2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{vmatrix}, \tag{3.2.2.5}$$

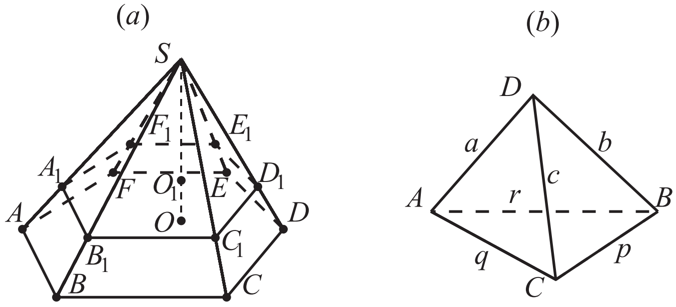

Figure 3.29. The pyramid cut by a plane and the original pyramid (*a*). A triangular pyramid (*b*).

where the right-hand side contains a determinant.

A pyramid is said to be *regular* if its base is a regular n-gon and the altitude passes through the center of the base. The altitude (issuing from the apex) of a lateral face is called the *apothem* of a regular pyramid. For a regular pyramid, the lateral surface area is

$$S_{\text{lat}} = \frac{1}{2}Pl, \tag{3.2.2.6}$$

where P is the perimeter of the base and l is the apothem.

2°. If a pyramid is cut by a plane parallel to the base, then it splits into two parts, a pyramid similar to the original pyramid and the *frustum* (Fig. 3.30*a*). The volume of the frustum is

$$V = \frac{1}{3}h(S_1 + S_2 + \sqrt{S_1 S_2}) = \frac{1}{3}hS_2\left[1 + \frac{a}{A} + \frac{a^2}{A^2}\right], \tag{3.2.2.7}$$

where S_1 and S_2 are the areas of the bases, a and A are two respective sides of the bases, and h is the altitude (the perpendicular distance between the bases).

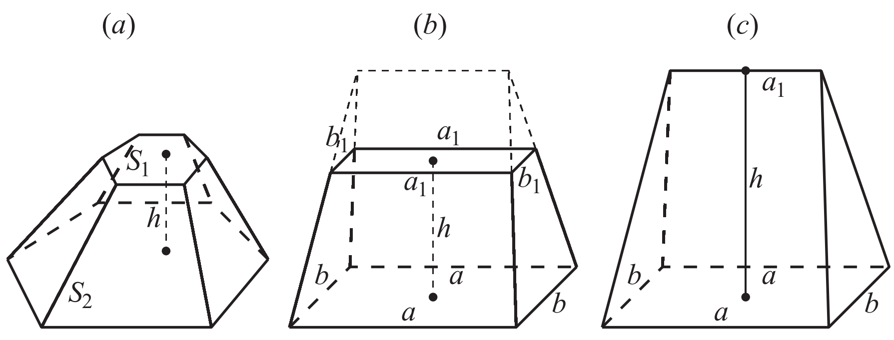

Figure 3.30. A frustum of a pyramid (*a*), an obelisk (*b*), and a wedge (*c*).

For a regular frustum, the lateral surface area is

$$S_{\text{lat}} = \frac{1}{2}(P_1 + P_2)l, \tag{3.2.2.8}$$

where P_1 and P_2 are the perimeters of the bases and l is the altitude of the lateral face.

3°. A hexahedron whose bases are rectangles lying in parallel planes and whose lateral faces form equal angles with the base, but do not meet at a single point, is called an *obelisk* (Fig. 3.30*b*). If a, b and a_1, b_1 are the sides of the bases and h is the altitude, then the volume of the hexahedron is

$$V = \frac{h}{6}[(2a + a_1)b + (2a_1 + a)b_1]. \tag{3.2.2.9}$$

4°. A pentahedron whose base is a rectangle and whose lateral faces are isosceles triangles and isosceles trapezoids is called a *wedge* (Fig. 3.30*c*). The volume of the wedge is

$$V = \frac{h}{6}(2a + a_1)b. \tag{3.2.2.10}$$

3.2.2-4. Regular polyhedra.

A polyhedron is said to be *regular* if all of its faces are equal regular polygons and all polyhedral angles are equal to each other. There exist five regular polyhedra (Fig. 3.31), whose properties are given in Table 3.4.

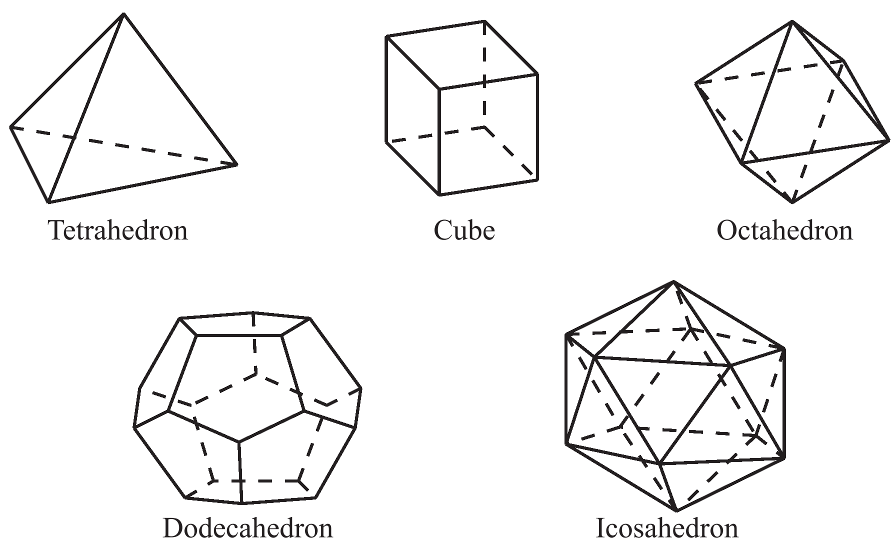

Figure 3.31. Five regular polyhedra.

3.2.3. Solids Formed by Revolution of Lines

3.2.3-1. Cylinder.

A *cylindrical surface* is a surface in space swept by a straight line (the *generator*) moving parallel to a given direction along some curve (the *directrix*) (Fig. 3.32*a*).

1°. A solid bounded by a closed cylindrical surface and two planes is called a *cylinder*; the planes are called the *bases* of the cylinder (Fig. 3.32*b*).

If P is the perimeter of the base, P_{sec} is the perimeter of the section perpendicular to the generator, S_{sec} is the area of this section, S_{bas} is the area of the base, and l is the length of

TABLE 3.4
Regular polyhedra (a is the edge length)

No.	Name	Number of faces and its shape	Number of vertices	Number of edges	Total surface area	Volume
1	Tetrahedron	4 triangles	4	6	$a^2\sqrt{3}$	$\dfrac{a^3\sqrt{2}}{12}$
2	Cube	6 squares	8	12	$6a^2$	a^3
3	Octahedron	8 triangles	6	12	$2a^2\sqrt{3}$	$\dfrac{a^3\sqrt{2}}{3}$
4	Dodecahedron	12 pentagons	20	30	$3a^2\sqrt{25+10\sqrt{5}}$	$\dfrac{a^3}{4}(15+7\sqrt{5})$
5	Icosahedron	20 triangles	12	30	$3a^2\sqrt{3}$	$\dfrac{5a^3}{12}(3+\sqrt{5})$

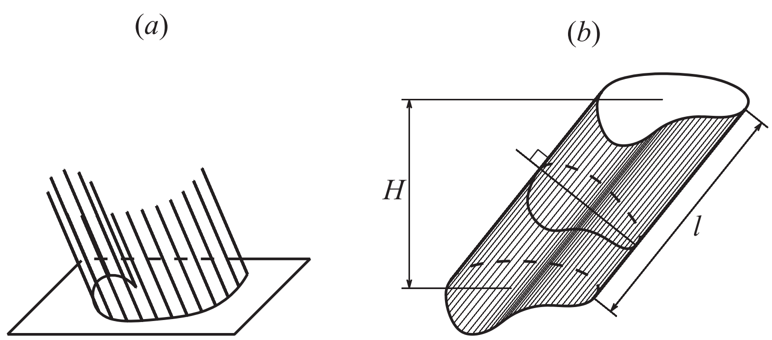

Figure 3.32. A cylindrical surface (a). A cylinder (b).

the generator, then the lateral surface area S_{lat} and the volume V of the cylinder are given by the formulas

$$
\begin{aligned}
S_{\text{lat}} &= PH = P_{\text{sec}}\,l,\\
V &= S_{\text{bas}}H = S_{\text{sec}}\,l.
\end{aligned}
\tag{3.2.3.1}
$$

In a *right* cylinder, the bases are perpendicular to the generator. In particular, if the bases are disks, then one speaks of a *right circular cylinder*. The volume, the lateral surface area, and the total surface area of a right circular cylinder are given by the formulas

$$
\begin{aligned}
V &= \pi R^2 H,\\
S_{\text{lat}} &= 2\pi R H,\\
S &= 2\pi R(R+H),
\end{aligned}
\tag{3.2.3.2}
$$

where R is the radius of the base.

A right circular cylinder is also called a *round cylinder*, or simply a *cylinder*.

2°. The part of a cylinder cut by a plane nonparallel to the base is called a *truncated cylinder* (Fig. 3.33*a*).

The volume, the lateral surface area, and the total surface area of a truncated cylinder

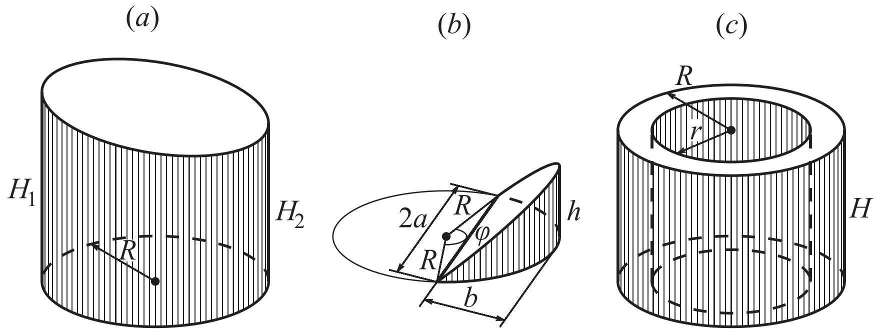

Figure 3.33. A truncated cylinder (a), a "hoof" (b), and a cylindrical tube (c).

are given by the formulas

$$\begin{aligned} V &= \pi R^2 \frac{H_1 + H_2}{2}, \\ S_{\text{lat}} &= \pi R(H_1 + H_2), \\ S &= \pi R\left[H_1 + H_2 + R + \sqrt{R^2 + \left(\frac{H_2 - H_1}{2}\right)^2}\,\right], \end{aligned} \tag{3.2.3.3}$$

where H_1 and H_2 are the maximal and minimal generators.

3°. A *segment of a round cylinder (a "hoof")* is a portion of the cylinder cut by a plane that is nonparallel to the base and intersects it. If R is the radius of the cylindrical segment, h is the height of the "hoof," and b is its width (for the other notation, see Fig. 3.33b), then the volume V and the lateral surface area S_{lat} of the "hoof" can be determined by the formulas

$$\begin{aligned} V &= \frac{h}{3b}\left[a(3R^2 - a^2) + 3R^2(b - R)\alpha\right] = \frac{hR^3}{b}\left(\sin\alpha - \frac{\sin^3\alpha}{3} - \alpha\cos\alpha\right), \\ S_{\text{lat}} &= \frac{2\pi R}{b}[(b - R)\alpha + a], \end{aligned} \tag{3.2.3.4}$$

where $\alpha = \frac{1}{2}\varphi$ is measured in radians.

4°. A solid bounded by two closed cylindrical surfaces and two planes is called a *cylindrical tube*; the planes are called the bases of the tube. The volume of a round cylindrical tube (Fig. 3.33c) is

$$V = \pi H(R^2 - r^2) = \pi H\delta(2R - r) = \pi H\delta(2r + \delta) = 2\pi H\delta\rho, \tag{3.2.3.5}$$

where R and r are the outer and inner radii, $\delta = R - r$ is the thickness, $\rho = \frac{1}{2}(R + r)$ is the midradius, and H is the height of the pipe.

3.2.3-2. Conical surface. Cone. Frustum of cone.

A *conical surface* is the union of straight lines (*generators*) passing through a fixed point (the *apex*) in space and any point of some space curve (the *directrix*) (Fig. 3.34a).

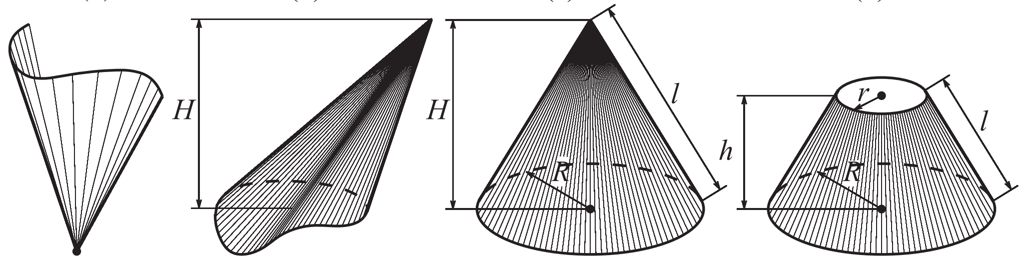

Figure 3.34. Conical surface (*a*). A cone (*b*), a right circular cone (*c*), and a frustum of a cone (*d*).

1°. A solid bounded by a conical surface with closed directrix and a plane is called a *cone*; the plane is the base of the cone (Fig. 3.34*b*). The volume of an arbitrary cone is given by the formula

$$V = \frac{1}{3} H S_{\text{bas}}, \tag{3.2.3.6}$$

where H is the altitude of the cone and S_{bas} is the area of the base.

A *right circular cone* (Fig. 3.34*c*) has a disk as the base, and its vertex is projected onto the center of the disk. If l is the length of the generator and R is the radius of the base, then the volume, the lateral surface area, and the total surface area of the right circular cone are given by the formulas

$$\begin{aligned} V &= \frac{1}{3}\pi R^2 H, \\ S_{\text{lat}} &= \pi R l = \pi R\sqrt{R^2 + H^2}, \\ S &= \pi R(R + l). \end{aligned} \tag{3.2.3.7}$$

2°. If a cone is cut by a plane parallel to the base, then we obtain a *frustum of a cone* (Fig. 3.34*d*). The length l of the generator, the volume V, the lateral surface area S_{lat}, and the total surface area S of the frustum of a right circular cone are given by the formulas

$$\begin{aligned} l &= \sqrt{h^2 + (R - r)^2}, \\ V &= \frac{\pi h}{3}(R^2 + r^2 + Rr), \\ S_{\text{lat}} &= \pi l(R + r), \\ S &= \pi[l(R + r) + R^2 + r^2], \end{aligned} \tag{3.2.3.8}$$

where r is the radius of the upper base and h is the altitude of the frustum of a cone.

3.2.3-3. Sphere. Spherical parts. Torus.

1°. The *sphere of radius R centered at O* is the set of points in space at the distance R from the point O (Fig. 3.35*a*). A solid bounded by a sphere is called a *ball*. Any section of the sphere by a plane is a circle. The section of the sphere by a plane passing through its center is called a *great circle* of radius R. There exists exactly one great circle passing through two arbitrary points on the sphere that are not antipodal (i.e., are not the opposite endpoints of a diameter). The smaller arc of this great circle is the shortest distance on the sphere between these points. Concerning the geometry of the sphere, see Section 3.3. The surface area S of the sphere and the volume V of the ball bounded by the sphere are given by the formulas

$$S = 4\pi R = \pi D^2 = \sqrt[3]{36\pi V^2},$$
$$V = \frac{4\pi R^3}{3} = \frac{\pi D^3}{6} = \frac{1}{6}\sqrt{\frac{S^3}{\pi}}, \qquad (3.2.3.9)$$

where $D = 2R$ is the *diameter* of the sphere.

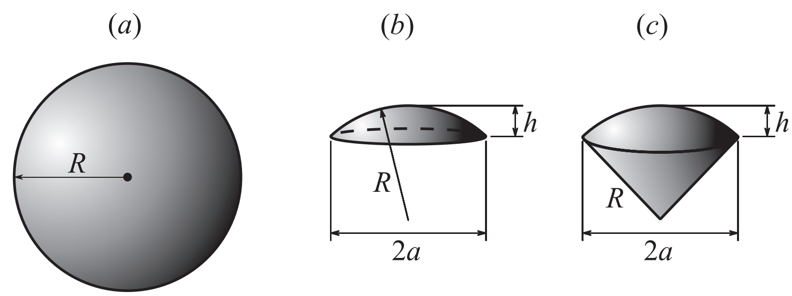

Figure 3.35. A sphere (a), a spherical cap (b), and a spherical sector (c).

2°. A portion of a ball cut from it by a plane is called a *spherical cap* (Fig. 3.35b). The width a, the area $S_{\rm lat}$ of the curved surface, the total surface area S, and the volume V of a spherical cap can be found from the formulas

$$a^2 = h(2R - h),$$
$$S_{\rm lat} = 2\pi Rh = \pi(a^2 + h^2),$$
$$S = S_{\rm lat} + \pi a^2 = \pi(2Rh + a^2) = \pi(h^2 + 2a^2), \qquad (3.2.3.10)$$
$$V = \frac{\pi h}{6}(3a^2 + h^2) = \frac{\pi h^2}{3}(3R - h),$$

where R and h are the radius and the height of the spherical cap.

3°. A portion of a ball bounded by the curved surface of a spherical cap and the conical surface whose base is the base of the cap and whose vertex is the center of the ball is called a *spherical sector* (Fig. 3.35c). The total surface area S and the volume V of a spherical sector are given by the formulas

$$S = \pi R(2h + a),$$
$$V = \frac{2}{3}\pi R^2 h, \qquad (3.2.3.11)$$

where a is the width, h is the height, and R is the radius of the sector.

4°. A portion of a ball contained between two parallel plane secants is called a *spherical segment* (Fig. 3.36a). The curved surface of a spherical segment is called a *spherical zone*, and the plane circular surfaces are the *bases* of a spherical segment. The radius R of the ball, the radii a and b of the bases, and the height h of a spherical segment satisfy the relation

$$R^2 = a^2 + \left(\frac{a^2 - b^2 - h^2}{2h}\right)^2. \qquad (3.2.3.12)$$

The curved surface area S_{lat}, the total surface area S, and the volume V of a spherical segment are given by the formulas

$$
\begin{aligned}
S_{\text{lat}} &= 2\pi Rh,\\
S &= S_{\text{lat}} + \pi(a^2 + b^2) = \pi(2Rh + a^2 + b^2),\\
V &= \frac{\pi h}{6}(3a^2 + 3b^2 + h^2).
\end{aligned}
\tag{3.2.3.13}
$$

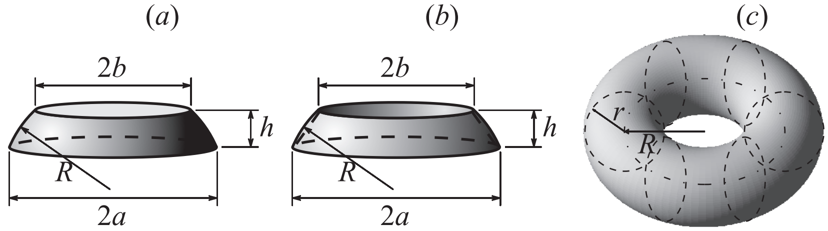

Figure 3.36. A spherical segment (*a*) and a spherical segment without the truncated cone inscribed in it (*b*). A torus (*c*).

If V_1 is the volume of the truncated cone inscribed in a spherical segment (Fig. 3.36*b*) and l is the length of its generator, then

$$V - V_1 = \frac{\pi h l^2}{6}. \tag{3.2.3.14}$$

4°. A *torus* is a surface generated by revolving a circle about an axis coplanar with the circle but not intersecting it. If the directrix is a circle (Fig. 3.36*c*), the radius R of the directrix is not less than the radius r of the generating circle ($R \geq r$), and the center of the generator moves along the directrix, then the surface area and the volume of the torus are given by the formulas

$$
\begin{aligned}
S &= 4\pi^2 Rr = \pi^2 Dd,\\
V &= 2\pi^2 Rr^2 = \frac{\pi^2 Dd^2}{4},
\end{aligned}
\tag{3.2.3.15}
$$

where $D = 2R$ and $d = 2r$ are the diameters of the generator and the directrix.

3.3. Spherical Trigonometry

3.3.1. Spherical Geometry

3.3.1-1. Great circle.

A *great circle* is a section of a sphere by a plane passing through the center.

Properties of great circles:

1. The radius of a great circle is equal to the radius of the sphere.
2. There is only one great circle through two arbitrary points that are not the opposite endpoints of a diameter.

The smaller arc of the great circle through two given points is called a *geodesic*, and the length of this arc is the *shortest distance* on the sphere between the two points. The great circles on the sphere play a role similar to the role of straight lines on the plane.

Any two points on the sphere determine a pencil of planes. The intersection of each plane in the pencil with the sphere is a circle. If two points are not the opposite endpoints of a diameter, then the plane passing through the center of the sphere determines the largest circle in the pencil, which is a great circle. The other circles are called *small circles*; the intersection with the sphere of the plane perpendicular to the plane containing the great circle is the smallest circle.

3.3.1-2. Measurement of arcs and angles on sphere. Spherical biangles.

The distances on the sphere are measured along great circle arcs. The great circle arc length between points A and B is given by the relation

$$\smile{AB} = R\alpha, \tag{3.3.1.1}$$

where R is the radius of the sphere and α is the corresponding central angle (in radians). If only the unit sphere (the radius $R = 1$) is considered, then each great circle arc can be characterized by the corresponding central angle (in radians). The angle between two intersecting great circle arcs is measured by the linear angle between the tangents to the great circles at the point of intersection or, which is the same, by the dihedral angle between the planes of the great circles.

Two intersecting great circles on the sphere form four *spherical biangles*. The area of a spherical biangle with the angle α is given by the formula

$$S = 2R^2\alpha. \tag{3.3.1.2}$$

3.3.2. Spherical Triangles

3.3.2-1. Basic notions and properties.

A figure formed by three great circle arcs pairwise connecting three arbitrary points on the sphere is called a *spherical triangle* (Fig. 3.37*a*). The *vertices* of a spherical triangle are the points of intersection of three rays issuing from the center of the sphere with the sphere. The angles less than π between the rays are called the sides a, b, and c of a spherical triangle. Such spherical triangles are called *Euler triangles*. To each side of a triangle there corresponds a great circle arc on the sphere. The angles α, β, and γ opposite the sides a, b, and c of a spherical triangle are the angles between the great circle arcs corresponding to the sides of the triangle, or, equivalently, the angles between the planes determined by these rays.

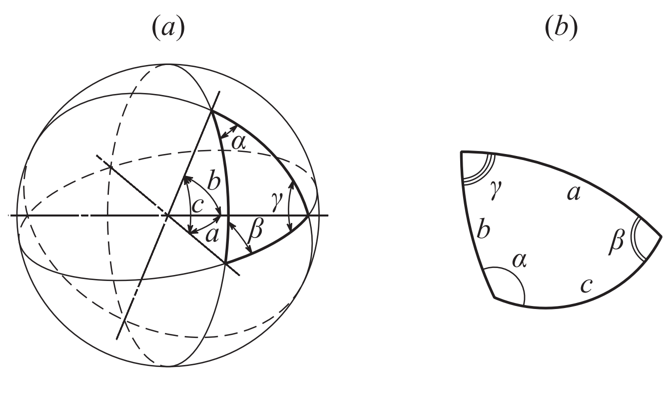

Figure 3.37. A spherical triangle.

By analogy with the circumcircle of a plane triangle, there is a "circumscribed cone of revolution" that contains the three straight lines determining the triangle; the axis of this cone is the intersection of the planes perpendicular to the sides at their midpoints. There also exists an "inscribed cone of revolution" that is tangent to the three planes corresponding to the spherical triangle; the axis of this cone is the intersection of the angle bisector planes. The "circumradius" $\overline{R}$ and the "inradius" r are defined as the angles equal to half the angles at the vertices of the first and the second cone, respectively.

If R is the radius of the sphere, then the area S of the spherical triangle is given by the formula

$$S = R^2\varepsilon, \tag{3.3.2.1}$$

where ε is the *spherical excess* defined as

$$\varepsilon = \alpha + \beta + \gamma - \pi \tag{3.3.2.2}$$

and measured in radians.

A spherical triangle is uniquely determined (up to a symmetry transformation) by:

1. Three sides.
2. Three angles.
3. Two sides and their included angle.
4. Two angles and their included side.

Let α, β, and γ be the angles and a, b, and c the sides opposite these angles in a spherical triangle (Fig. 3.37*b*). Table 3.5 presents the basic properties and relations characterizing spherical triangles (with the notation $2p = a + b + c$ and $2P = \alpha + \beta + \gamma - \pi$). From the relations given in Table 3.5, one can derive all missing relations by cyclically permuting the sides a, b, and c and the angles α, β, and γ.

LEGENDRE'S THEOREM. *The area of a spherical triangle with small sides (i.e., with sides that are small compared with the radius of the sphere) is approximately equal to the area of a plane triangle with the same sides; the difference between each angle of the plane triangle and the corresponding angle of the spherical triangle is approximately equal to one-third of the spherical excess.*

The law of sines, the law of cosines, and the half-angle theorem in spherical trigonometry for small sides become the corresponding theorems of the linear (plane) trigonometry.

Table 3.6 allows one to find the sides and angles of an arbitrary spherical triangle if three appropriately chosen sides and/or angles are given.

3.3.2-2. Rectangular spherical triangle.

A spherical triangle is said to be *rectangular* if at least one of its angles, for example, γ, is equal to $\frac{1}{2}\pi$ (Fig. 3.38*a*); the opposite side c is called the *hypotenuse*.

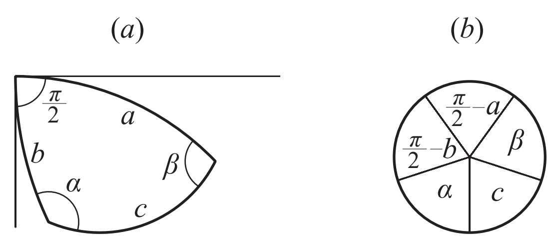

Figure 3.38. A rectangular spherical triangle (*a*). The Neper rules (*b*).

TABLE 3.5
Basic properties and relations characterizing spherical triangles

No.	The name of property	Properties and relations
1	Triangle inequality	The sum of lengths of two sides is greater than the length of the third side. The absolute value of the difference between the lengths of two sides is less than the length of the third side, $a+b>c, \quad \lvert a-b\rvert<c$
2	Sum of two angles of a triangle	The sum of two angles of a triangle is greater than the third angle increased by π, $\alpha+\beta<\pi+\gamma$
3	The greatest side and the greatest angle	The greatest side is opposite the greatest angle, $a<b$ if $\alpha<\beta$; $a=b$ if $\alpha=\beta$
4	Sum of angles of a triangle	The sum of the angles lies between π and 3π, $\pi<\alpha+\beta+\gamma<3\pi$
5	Sum of sides of a triangle	The sum of sides lies between 0 and 2π $0<a+b+c<2\pi$
6	The law of sines	$\dfrac{\sin a}{\sin\alpha}=\dfrac{\sin b}{\sin\beta}=\dfrac{\sin c}{\sin\gamma}$
7	The law of cosines of sides	$\cos c=\cos a\cos b+\sin a\sin b\cos\gamma$
8	The law of cosines of angles	$\cos\gamma=-\cos\alpha\cos\beta+\sin\alpha\sin\beta\cos c$
9	Half-angle formulas	$\sin\dfrac{\gamma}{2}=\sqrt{\dfrac{\sin(p-a)\sin(p-b)}{\sin a\sin b}}, \quad \cos\dfrac{\gamma}{2}=\sqrt{\dfrac{\sin p\sin(p-c)}{\sin a\sin b}},$ $\tan\dfrac{\gamma}{2}=\sqrt{\dfrac{\sin(p-a)\sin(p-b)}{\sin p\sin(p-c)}}$
10	Half-side theorem	$\sin\dfrac{c}{2}=\sqrt{\dfrac{-\sin P\sin(P-\gamma)}{\sin\alpha\sin\beta}}, \quad \cos\dfrac{c}{2}=\sqrt{\dfrac{\sin(P-\alpha)\sin(P-\beta)}{\sin\alpha\sin\beta}},$ $\tan\dfrac{c}{2}=\sqrt{\dfrac{-\sin P\sin(P-\gamma)}{\sin(P-\alpha)\sin(P-\beta)}}$
11	Neper's analogs	$\tan\dfrac{c}{2}\cos\dfrac{\alpha-\beta}{2}=\tan\dfrac{a+b}{2}\cos\dfrac{\alpha+\beta}{2},$ $\tan\dfrac{c}{2}\sin\dfrac{\alpha-\beta}{2}=\tan\dfrac{a-b}{2}\sin\dfrac{\alpha+\beta}{2},$ $\cot\dfrac{\gamma}{2}\cos\dfrac{a-b}{2}=\tan\dfrac{\alpha+\beta}{2}\cos\dfrac{a+b}{2},$ $\cot\dfrac{\gamma}{2}\sin\dfrac{a-b}{2}=\tan\dfrac{\alpha-\beta}{2}\sin\dfrac{a+b}{2}$
12	D'Alembert (Gauss) formulas	$\sin\dfrac{\gamma}{2}\sin\dfrac{a+b}{2}=\sin\dfrac{c}{2}\cos\dfrac{\alpha-\beta}{2}, \quad \sin\dfrac{\gamma}{2}\sin\dfrac{a+b}{2}=\cos\dfrac{c}{2}\cos\dfrac{\alpha+\beta}{2},$ $\cos\dfrac{\gamma}{2}\sin\dfrac{a-b}{2}=\sin\dfrac{c}{2}\sin\dfrac{\alpha-\beta}{2}, \quad \cos\dfrac{\gamma}{2}\cos\dfrac{a-b}{2}=\cos\dfrac{c}{2}\sin\dfrac{\alpha+\beta}{2}$
13	Product formulas	$\sin a\cos\beta=\cos b\sin c-\cos\alpha\sin b\cos c,$ $\sin a\cos b=\cos\beta\sin c-\cos a\sin\beta\cos\gamma$
14	The "circumradius" $\overline{R}$	$\cot\overline{R}=\sqrt{\dfrac{\sin(P-\alpha)\sin(P-\beta)\sin(P-\gamma)}{\sin P}}=\cot\dfrac{\alpha}{2}\sin(\alpha-P)$
15	The "inradius" r	$\tan r=\sqrt{\dfrac{\sin(p-\alpha)\sin(p-\beta)\sin(p-\gamma)}{\sin p}}=\tan\dfrac{\alpha}{2}\sin(p-\alpha)$

TABLE 3.5 (*continued*)
Basic properties and relations characterizing spherical triangles

No.	The name of property	Properties and relations
16	Willier's formula for the spherical excess ε	$\tan\frac{P}{2} = \tan\frac{\varepsilon}{4} = \sqrt{\tan\frac{p}{2}\tan\frac{p-a}{2}\tan\frac{p-b}{2}\tan\frac{p-c}{2}}$
17	L'Huiller equation	$\tan\left(\frac{\gamma}{2}-\frac{\varepsilon}{4}\right) = \sqrt{\frac{\tan\frac{p-a}{2}\tan\frac{p-b}{2}}{\tan\frac{p}{2}\tan\frac{p-c}{2}}}$

TABLE 3.6.
Solution of spherical triangles

No.	Three parts specified	Formulas for the remaining parts
1	Three sides a, b, c	The angles α, β, and γ are determined by the half-angle formulas and the cyclic permutation. Remark. $0<a+b+c<2\pi$. The sum and difference of two sides are greater than the third.
2	Three angles α, β, γ	The sides a, b, and c are determined by the half-side theorems and the cyclic permutation. Remark. $\pi<\alpha+\beta+\gamma<3\pi$. The sum of two angles is less than π plus the third angle.
3	Two sides a, b and the included angle γ	*First method.* $\alpha+\beta$ and $\alpha-\beta$ are determined from Neper's analogs, then α and β can be found; side a is determined from the law of cosines, $\sin c = \sin\gamma\frac{\sin a}{\sin\alpha}$. *Second method.* The law of cosines of sides is applied, $\cos c = \cos a\cos b + \sin a\sin b\cos\gamma$, $\cos\beta = \frac{\cos b - \sin a\sin c}{\sin a\sin c}$, $\cos\alpha = \frac{\cos a - \sin b\sin c}{\sin b\sin c}$. Remark 1. If $\gamma > \beta$ ($\gamma < \beta$), then c must be chosen so that $c > b$ ($c < b$). Remark 2. The quantities c, α, and β are determined uniquely.
4	A side c and the two angles α, β adjacent to it	*First method.* $a+b$ and $a-b$ are determined from Neper's analogs, then a and b can be found; angle γ is determined from the law of sines, $\sin\gamma = \sin c\frac{\sin\alpha}{\sin a}$. *Second method.* The law of cosines of angles is applied, $\cos\gamma = -\cos\alpha\cos\beta + \sin\alpha\sin\beta\cos c$, $\cos a = \frac{\cos\alpha + \cos\beta\cos\gamma}{\sin\beta\sin\gamma}$, $\cos b = \frac{\cos\beta + \cos\alpha\cos\gamma}{\sin\alpha\sin\gamma}$. Remark 1. If $c > b$ ($c < b$), then γ must be chosen so that $\gamma > \beta$ ($\gamma < \beta$). Remark 2. The quantities γ, a, and b are determined uniquely.
5	Two sides a, b and the angle α opposite one of them	β is determined by the law of sines, $\sin\beta = \sin\alpha\frac{\sin b}{\sin a}$. The elements c and γ can be found from Neper's analogs. Remark 1.The problem has a solution for $\sin b\sin\alpha \le \sin a$. Remark 2. Different cases are possible: 1. If $\sin a \ge \sin b$, then the solution is determined uniquely. 2. If $\sin b\sin\alpha < \sin a$, then there are two solutions β_1 and β_2, $\beta_1+\beta_2=\pi$. 3. If $\sin b\sin\alpha = \sin a$, then the solution is unique: $\beta = \frac{1}{2}\pi$.
6	Two angles α, β and the side a opposite one of them	b is determined by the law of sines, $\sin b = \sin a\frac{\sin\beta}{\sin\alpha}$. The elements c and γ can be found from Neper's analogs. Remark 1. The problem has a solution for $\sin a\sin\beta \le \sin\alpha$. Remark 2. Different cases are possible: 1. If $\sin\alpha \ge \sin\beta$, then the solution is determined uniquely. 2. If $\sin\beta\sin\alpha < \sin a$, then there are two solutions b_1 and b_2, $b_1+b_2=\pi$. 3. If $\sin\beta\sin\alpha = \sin a$, then the solution is unique: $b = \frac{1}{2}\pi$.

The following basic relations hold for spherical triangles:

$$\begin{aligned}
&\sin a = \cos\left(\tfrac{\pi}{2} - a\right) = \sin\alpha\sin c = \cot\left(\tfrac{\pi}{2} - b\right)\cot\beta = \tan b\cot\beta,\\
&\sin b = \cos\left(\tfrac{\pi}{2} - b\right) = \sin\beta\sin c = \cot\left(\tfrac{\pi}{2} - a\right)\cot\alpha = \tan a\cot\alpha,\\
&\cos c = \sin\left(\tfrac{\pi}{2} - a\right)\sin\left(\tfrac{\pi}{2} - b\right) = \cos a\cos b = \cot\alpha\cot\beta,\\
&\cos\alpha = \sin\left(\tfrac{\pi}{2} - a\right)\sin\beta = \cos a\sin\beta = \cot\left(\tfrac{\pi}{2} - b\right)\cot c = \tan b\cot c,\\
&\cos\beta = \sin\left(\tfrac{\pi}{2} - b\right)\sin\alpha = \cos b\sin\alpha = \cot\left(\tfrac{\pi}{2} - a\right)\cot c = \tan a\cot c,
\end{aligned} \tag{3.3.2.3}$$

which can be obtained from the *Neper rules*: if the five parts of a spherical triangle (the right angle being omitted) are written in the form of a circle in the order in which they appear in the triangle and the legs a and b are replaced by their complements to $\frac{1}{2}\pi$ (Fig. 3.38*b*), then the cosine of each part is equal to the product of sines of the two parts not adjacent to it, as well as to the product of the cotangents of the two parts adjacent to it.

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